that the optimal body has, close to the stagnation point, a conical shape with apex angle of 120°.

We note that the terms discarded in Eqs. (4.6) are of the order $\rho^2 R$ and, therefore, in satisfying the inequality $\rho^2 R \ll 1$ ($\rho^3 R \ll 1$ in the axially symmetric case) we can, with a high degree of accuracy, assume the flow to be Stokes flow. Therefore, in the plane-parallel case, and also in the axially symmetric case, the magnitude of the angle θ_1 depends neither on the Reynolds number nor on whether the singular point is at the front or at the back.

In conclusion, the author thanks F. L. Chernous'ko for his statement of the problem and N. V. Banichuk for useful discussions.

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PROBLEM WITH DISCONTINUOUS BOUNDARY CONDITIONS AND THE DIFFUSION BOUNDARY LAYER APPROXIMATION

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We consider the stationary convective diffusion problem (heat conductivity problem) which occurs in the flow of a fluid with a shear velocity profile above an infinite plate. On the plate we assume discontinuous boundary conditions of zero flow, zero concentration type. This problem is solved by use of the Wiener-Hopf method with longitudinal diffusion taken into account. We obtain the exact solution in the form of a complex integral and we determine an asymptotic expansion for the density of the flow on the plate close to and far from a discontinuity point in the boundary conditions. We show that close to this point the diffusion boundary layer approximation (DBLA) is unsuitable. We determine the character of the singularity in the flow density at the discontinuity point and we make corrections to the DBLA.

1. Statement of the problem and the Wiener-Hopf method. The mathematical statement of our problem is the following:

$$2Vy\frac{\partial C}{\partial x} = \frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2}, \quad 0 < x < \infty, \quad y > 0, \quad V > 0 \tag{1.1}$$

$$\frac{\partial C}{\partial y}(x, 0) = 0, \quad x < 0; \quad C(x, 0) = 0, \quad x > 0 \tag{1.2}$$

$$C(x, y) \to 1, \quad x \to -\infty \text{ or } y \to \infty$$

We seek a bounded solution of this problem. All variables are assumed to be dimensionless.

The behavior of the solution as $x \to -\infty$ makes a direct application of the Fourier transformation difficult. To avoid this difficulty we consider the more general equation

$$2(Vy + v)\frac{\partial C}{\partial x} = \frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2}, \quad v > 0 \tag{1.3}$$

The usefulness of this generalization will become clear as we proceed. We introduce the new dependent variable

$$\varphi(x, y) = e^{-vx}[1 - C(x, y)] \tag{1.4}$$

We can obtain the following estimate for the solution of Eq. (1.3) with the boundary conditions (1.2):

$$1 - C(x, y) \leqslant Ae^{2vx}, \quad x \to -\infty \tag{1.5}$$

This estimate guarantees existence of the Fourier transformation (in the classical sense) for the function $\varphi(x, y)$. A quick derivation of this estimate follows from Eq. (1.3) by setting $V = 0$ and discarding, as $x \to -\infty$, the term $\partial^2 C / \partial y^2$. It is clear that increasing V only strengthens the inequailty (1.5). On the other hand, we shall solve the problem (1.3) with $V = 0$ exactly (see Eq. (2.9)) and thereby confirm the estimate (1.5) directly. Thus,

$$\varphi(x, y) \leqslant \begin{cases} Ae^{vx}, & x \to -\infty \\ Be^{-vx}, & x \to +\infty \end{cases} \tag{1.6}$$

To determine $\varphi(x, y)$ we solve the problem

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} - [v^2 + 2Vvy]\varphi = 2Vy\frac{\partial \varphi}{\partial x} \tag{1.7}$$

$$\frac{\partial \varphi}{\partial y}(x, 0) = 0, \quad x < 0; \quad \varphi(x, 0) = e^{-vx}, \quad x > 0$$

$$\varphi \to 0, \qquad r = \sqrt{x^2 + y^2} \to \infty$$

For the solution of problem (1.7) we employ the complex Fourier transformation

$$\Phi(\alpha, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{i\alpha x}\varphi(x, y)\, dx, \quad \alpha = \sigma + i\tau$$

The estimate (1.6) ensures the analyticity of $\Phi(\alpha, y)$ in the strip $-v < \tau < v$. The form of the inverse transformation is

$$\varphi(x, y) = \frac{1}{\sqrt{2\pi}} \int_{i\tau-\infty}^{i\tau+\infty} e^{-i\alpha x}\Phi(\alpha, y)\, d\alpha, \quad -v < \tau < v \tag{1.8}$$

For $\Phi(\alpha, y)$ we obtain the following equation:

$$d^2\Phi/dy^2 + [2iVy(\alpha + iv) - \alpha^2 - v^2]\Phi = 0$$

The solution of this equation, a decreasing function for $y \to +\infty$ may be expressed in terms of the Airy function

$$\Phi(\alpha, y) = A(\alpha)\mathrm{Ai}[h(y)] \tag{1.9}$$

$$\mathrm{Ai}(z) = \frac{1}{\pi}\int_0^\infty \cos\left(sz + \frac{1}{3}s^3\right) ds, \quad h(y) = \exp\left(-\frac{2\pi i}{3}\right)\frac{2iV(\alpha + iv)y - \alpha^2 - v^2}{[2V(\alpha + iv)]^{2/3}}$$

The function $A(\alpha)$ is, for the present, arbitrary. For the Airy function we employ the normalization used in [1]. We assume that $-\pi < \arg\alpha < \pi$, and that the branches of the functions $(\alpha \pm iv)^r$ are fixed by the condition $(\alpha \pm iv)^r \to \sigma^r$ as $\sigma \to +\infty$ in the strip $-v < \tau < v$. We make cuts along the intervals $(iv, i\infty)$ and $(-i\infty, -iv)$ of the imaginary axis.

In the sequel we employ the following properties of the Airy function [1, 2].

1. The functions $\mathrm{Ai}(z)$ and $\mathrm{Ai}'(z) = d\mathrm{Ai}(z)/dz$ are entire functions, all of whose zeros in the z-plane are simple and lie on the negative real semi-axis.

2. The following asymptotic expansions are valid:

$$\mathrm{Ai}(z) = \frac{1}{2\sqrt{\pi}} \exp\left(-\frac{2}{3} z^{3/2}\right)\left[1 - \frac{5}{48} z^{-3/2} + O(z^{-3})\right], \quad -\pi < \arg z < \pi \quad (1.10)$$

$$\mathrm{Ai}(z) = \frac{1}{\sqrt{\pi}} \theta^{-1/4}\left[\sin\left(\frac{2}{3}\theta^{3/2} + \frac{\pi}{4}\right) - \frac{5}{48}\theta^{-3/2}\cos\left(\frac{2}{3}\theta^{3/2} + \frac{\pi}{4}\right) + O(\theta^{-3})\right], \quad z = -\theta, \quad \theta > 0$$

It can be verified that with the choice of branches indicated above, $\Phi(\alpha, y) \to 0$ for $y \to +\infty$ and $-v < \tau < v$.

We use the Wiener-Hopf method [3] to determine $A(\alpha)$. We introduce the notation

$$\Phi_+(\alpha, 0) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{i\alpha x}\varphi(x, 0)\,dx, \quad \Phi_-(\alpha, 0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 e^{i\alpha x}\varphi(x, 0)\,dx \quad (1.11)$$

$$\Phi_+'(\alpha, 0) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{i\alpha x}\frac{\partial\varphi}{\partial y}(x, 0)\,dx, \quad \Phi_-'(\alpha, 0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 e^{i\alpha x}\frac{\partial\varphi}{\partial y}(x, 0)\,dx$$

The assumption concerning the existence of the integral $\Phi_+'(\alpha, 0)$ corresponds to the assumption that the material flow density of the plate, namely, $j(x) = \partial C(x, 0)/\partial y$, has an integrable singularity as $x \to +0$. It follows from this that $\Phi_+'(\alpha, 0) \to 0$ as $\alpha \to \infty$. In fact [4], if $\partial\varphi(x, 0)/\partial y \sim Ax^\lambda$, $x \to +0$ and $-1 < \lambda < 0$, then

$$\Phi_+'(\alpha, 0) \sim A(2\pi)^{-1/2}\Gamma(\lambda + 1)\exp\left[\frac{\pi i}{2}(\lambda + 1)\right]\alpha^{-\lambda-1}, \quad \alpha \to \infty \quad (1.12)$$

(Concerning the assumption made on the nature of the singularity of $j(x)$ as $x \to +\infty$ see the note at the end of the paper).

Of the four functions (1.11) the functions $\Phi_-(\alpha, 0)$ and $\Phi_+'(\alpha, 0)$ are unknown. Taking into account the fact that

$$\Phi_+(\alpha, 0) + \Phi_-(\alpha, 0) = A(\alpha)\,\mathrm{Ai}\,[h(0)]$$

$$\Phi_+'(\alpha, 0) + \Phi_-'(\alpha, 0) = A(\alpha)\frac{d}{dy}\mathrm{Ai}\,[h(0)]$$

and eliminating $A(\alpha)$ from these equations, we obtain the Wiener-Hopf problem

$$\Phi_+'(\alpha, 0) + \Phi_-'(\alpha, 0) = f(\alpha)\;[\Phi_+(\alpha, 0) + \Phi_-(\alpha, 0)]$$

$$f(\alpha) = \exp\left(-\frac{i\pi}{6}\right)(2V)^{1/3}(\alpha + iv)^{1/3}\frac{\mathrm{Ai}'(z)}{\mathrm{Ai}(z)}, \quad z \equiv h(0) = \frac{\exp(i\pi/3)(\alpha^2 + v^2)}{(2V)^{2/3}(\alpha + iv)^{2/3}}$$

We assume that the problem of factorizing the function $f(\alpha)$ has been solved, i. e. we assume that a representation $f(\alpha) = f_+(\alpha) f_-(\alpha)$ has been found such that $f_+(\alpha)$ is analytic and has no zeros for $\tau > -v$, while $f_-(\alpha)$ is analytic and has no zeros for $\tau < v$. Then, following the Wiener-Hopf method [3], taking into account the fact that $\Phi_-'(\alpha, 0) = 0$ and $\Phi_+'(\alpha, 0) \to 0$ for $\alpha \to \infty$, we find

$$\Phi_+'(\alpha, 0) = G_+(\alpha) f_+(\alpha), \quad \Phi_-(\alpha, 0) = -\frac{1}{f_-(\alpha)} G_-(\alpha)$$

The functions $G_+(\alpha)$ and $G_-(\alpha)$ are analytic for $\tau > -v$ and $\tau < v$, respectively, and are determined from the condition

$$G(\alpha) \equiv f_-(\alpha)\Phi_+(\alpha, 0) = G_+(\alpha) + G_-(\alpha) \tag{1.13}$$

which yields the result [3]

$$G_+(\alpha) = \frac{1}{2\pi i} \int_{-\infty+ic_1}^{\infty+ic_1} \frac{G(t)}{t-\alpha} dt, \quad G_-(\alpha) = -\frac{1}{2\pi i} \int_{-\infty+ic_2}^{\infty+ic_2} \frac{G(t)}{t-\alpha} dt \tag{1.14}$$

$$(-v < c_1 < \tau < c_2 < v)$$

Taking note of the relation (1.13), we have

$$\Phi(\alpha, 0) = \Phi_+(\alpha, 0) + \Phi_-(\alpha, 0) = G_+(\alpha) / f_-(\alpha) \tag{1.15}$$

$$\Phi(\alpha, y) = \Phi(\alpha, 0) \frac{\mathrm{Ai}\,[h(y)]}{\mathrm{Ai}\,(z)} = \frac{G_+'(\alpha)\,\mathrm{Ai}\,[h(y)]}{\mathrm{Ai}\,(z)} \tag{1.16}$$

The quantity $\partial\varphi(x, 0) / \partial y$, appearing in the expression for $j(x)$ is determined directly in terms of $\Phi_+'(\alpha, 0)$

$$\frac{\partial\varphi}{\partial y}(x, 0) = \frac{1}{\sqrt{2\pi}} \int_{i\tau-\infty}^{i\tau+\infty} \Phi_+'(\alpha, 0) e^{-i\alpha x} d\alpha \tag{1.17}$$

Thus, solution of the problem reduces to factorizing the function $f(\alpha)$.

2. Factorization. We consider now the analytic properties of the function $f(\alpha)$. Using the first of the relations (1.10), we find

$$f(\alpha) = -\sqrt{\alpha^2 + v^2}\,[1 + {}^1/_4 z^{-3/2} + O(z^{-3})], \quad \mathrm{Im}\,(\alpha + iv) > 0 \tag{2.1}$$

We note that $f(\alpha) = -\sqrt{\alpha^2 + v^2}$ corresponds to the exact solution of the problem (1.7) for $V = 0$; this is to be expected, since if $V \to 0$ then $z \to \infty$. As has already been mentioned, all the zeros of the functions $\mathrm{Ai}\,(z)$ and $\mathrm{Ai}'\,(z)$ in the z-plane lie on the negative real semi-axis. Let $-r_k$ $(r_k > 0)$, $k = 1, 2, \ldots$ be the zeros of $\mathrm{Ai}\,(z)$, while $-r_k'$ $(r_k' > 0)$, $k = 0, 1, 2, \ldots$ be the zeros of $\mathrm{Ai}'\,(z)$. We consider now how the indexing of these zeros differs.

We introduce the quantities t_k and t_k'

$$r_k^{3/4} = \sqrt{\frac{3\pi}{2}}\, t_k, \quad r_k'^{3/4} = \sqrt{\frac{3\pi}{2}}\, t_k', \quad t_0' \equiv t$$

From the expressions (1.10) we easily obtain the asymptotic expansions

$$t_k = \sqrt{k}\left[1 - \frac{1}{8k} - \left(\frac{1}{128} - \frac{5}{144\pi^2}\right)\frac{1}{k^2} + O\left(\frac{1}{k^3}\right)\right]$$

$$t_k' = \sqrt{k}\left[1 + \frac{1}{8k} - \left(\frac{1}{128} + \frac{7}{144\pi^2}\right)\frac{1}{k^2} + O\left(\frac{1}{k^3}\right)\right]$$

Taking into account the manner in which the branches were chosen, we find that the poles α_k of the function $f(\alpha)$ which coincide with the zeros of $\mathrm{Ai}\,[z(\alpha)]$, are simple. They lie on the positive imaginary axis in the α-plane and are determined from the condition

$$\alpha_k = iv(s_k + 1)$$

where s_k is the positive root of the equation

$$r_k = (2V)^{2/3} (v)^{-4/3} s (s + 2)^{1/3}$$

The zeros α_k' of the function $f(\alpha)$ (also simple) lie on this same line

$$\alpha_k' = iv (s_k' + 1), \quad r_k' = (2V)^{2/3} (v)^{-4/3} (s' + 2)^{1/3} s'$$

In the case of greatest interest $\sqrt{2V}v^{-1} \gg 1$ we have

$$\alpha_k = ir_k^{3/4} \sqrt{2v} = i\sqrt{3\pi V t_k}, \quad \alpha_k' = i\sqrt{3\pi V t_k'} \tag{2.2}$$

The function $f(\alpha)$ has no other singularities in the upper half-plane. In the lower half-plane $f(\alpha)$ has a cut $(-i\infty, -iv)$ along the negative imaginary axis and has no zeros and poles. We introduce the function

$$g(\alpha) = -f(\alpha) / \sqrt{\alpha^2 + v^2} = -z^{-1/2} \operatorname{Ai}'(z) / \operatorname{Ai}(z) \tag{2.3}$$

Its expansions for large and small z (large and small α) are the following:

$$g(\alpha) = 1 + {}^1/_4 z^{-3/2} + O(z^{-3}) \tag{2.4}$$

$$g(\alpha) = Bz^{-1/2}\left[1 + Bz + \left(\frac{B^2}{2} - B\right) z^2 + O(z^3)\right]$$

$$B = -\frac{\operatorname{Ai}'(0)}{\operatorname{Ai}(0)} = \frac{3^{1/3}\Gamma(2/3)}{\Gamma(1/3)}$$

Noting the properties of $f(\alpha)$, we find that $g(\alpha)$ is analytic and has no zeros in the strip $-v < \tau < v$ and that $g(\alpha) \to 1$ as $\alpha \to \infty$ in this strip. Consequently, the function $g(\alpha)$ satisfies Theorem C of [3] and may be factored in the standard way [3]

$$g(\alpha) = g_+(\alpha) g_-(\alpha), \quad g_+(\alpha) = \exp h_+(\alpha), \quad g_-(\alpha) = \exp h_-(\alpha) \tag{2.5}$$

$$h_+(\alpha) = \frac{1}{2\pi i} \int_{ic'-\infty}^{ic'+\infty} \frac{\ln g(\xi)}{\xi - \alpha} d\xi, \quad h_-(\alpha) = -\frac{1}{2\pi i} \int_{ic''-\infty}^{ic''+\infty} \frac{\ln g(\xi)}{\xi - \alpha} d\xi \tag{2.6}$$

$$-v < c' < c_1 < \tau < c_2 < c'' < v$$

In Eqs. (2.6) the branch of the logarithm is chosen in such a way that $\ln g(\xi) \to 0$ as $\xi \to \infty$. The factorization of $f(\alpha)$ is now made without difficulty and we find

$$f_\pm(\alpha) = \mp \sqrt{\alpha \pm iv}\, g_\pm(\alpha)$$

In Eqs. (1.14) the integral for $G_+(\alpha)$ is readily calculated upon noting that

$$\Phi_+(\alpha, 0) = \frac{i}{\sqrt{2\pi}} \frac{1}{\alpha + iv}$$

the result is found to be

$$G_+(\alpha) = \frac{i}{\sqrt{2\pi}} \frac{f_-(-iv)}{\alpha + iv}$$

This enables us to express $\Phi(\alpha, 0)$ and $\Phi_+'(\alpha, 0)$ directly in terms of $g_+(\alpha)$ and $g_-(\alpha)$. Thus

$$\Phi_+'(\alpha, 0) = \frac{-i}{\sqrt{2\pi}} (-2iv)^{1/2} g_-(-iv) \frac{g_+(\alpha)}{(\alpha + iv)^{1/2}} \tag{2.7}$$

$$\Phi(\alpha, 0) = \frac{i}{\sqrt{2\pi}} \frac{(-2iv)^{1/2} g_-(-iv)}{(\alpha - iv)^{1/2}(\alpha + iv)\, g_-(\alpha)} \tag{2.8}$$

The expression $(-2iv)^{1/2}$ is here taken on the right side of the cut. The presence of this factor shows that in the passage to the limit $v \to 0$, we must proceed with caution. We note that the relations (1.8), (1.16), (2.5), (2.6) and (2.8) give a complete formal solution of the problem (1.7).

In the case $V = 0$, as Eqs. (2.1), (2.4) show, it is necessary to set $g_+(\alpha) = g_-(\alpha) = 1$ in the relations (2.7) and (2.8). Then, using these relations and the corresponding inversion formulas, we obtain

$$\varphi(x, 0) = \frac{1}{\sqrt{\pi}} e^{vx} \Gamma\left(\frac{1}{2}, \, -2vx\right), \quad x < 0, \quad V = 0 \tag{2.9}$$

$$\frac{\partial \varphi}{\partial y}(x, 0) = -\sqrt{\frac{2v}{x\pi}}\, e^{-xv}, \quad x > 0, \quad V = 0$$

If $-2vx \gg 1$ $(x < 0)$, then

$$\varphi(x, 0) = \frac{1}{\sqrt{\pi}} e^{vx} (-2vx)^{-1/2} \left[1 + O\left(\frac{1}{|vx|}\right)\right]$$

which confirms the estimate (1.6).

In the sequel, we assume that $\sqrt{2V} / v \gg 1$ and, in the explicit form of v, we shall retain only terms containing singularities as $v \to 0$.

3. Calculation of the flow density. We show that in the equations (2.6) the integral for $h_-(\alpha)$ can be evaluated exactly. The important factor is that function $\mathrm{Ai}'(z) / \mathrm{Ai}(z)$ appearing in $g(\alpha)$ has no branches in the upper half-plane α, and this determines the choice of $h_-(\alpha)$ for the computation. Differentiating $h_-(\alpha)$ with respect to α and integrating once by parts [5], we find

$$\frac{dh_-}{d\alpha} = -\frac{1}{2\pi i} \int_{ic''-\infty}^{ic''+\infty} \frac{1}{\xi - \alpha} \frac{1}{g} \frac{dg}{d\xi} d\xi =$$

$$\frac{1}{2\pi i} \int_{ic''-\infty}^{ic''+\infty} \frac{1}{\xi - \alpha} \frac{dz}{d\xi} \left[\frac{1}{2}\frac{1}{z} + \frac{\mathrm{Ai}'(z)}{\mathrm{Ai}(z)} - z\frac{\mathrm{Ai}(z)}{\mathrm{Ai}'(z)}\right]$$

$$\frac{dz}{d\xi} = \frac{d}{d\xi} \frac{\exp(i\pi/3)(\xi^2 + v^2)}{(2V)^{2/3}(\xi + iv)^{2/3}} = \frac{\exp(i\pi/3)}{(2V)^{2/3}} \left[\frac{2\xi}{\xi + iv} - \frac{2}{3} \frac{\xi^2 + v^2}{(\xi + iv)^{5/3}}\right]$$

It is immediately evident that in the last integral the integrand function has no branches in the upper half-plane. Closing the path of integration in the upper half-plane and following the type of reasoning used in the proof of Jordan's lemma [6], we obtain

$$\frac{dh_-}{d\alpha} = \frac{1}{2(iv - \alpha)} - \frac{1}{\alpha_0' - \alpha} + \sum_{k=1}^{\infty} \left(\frac{1}{\alpha_k - \alpha} - \frac{1}{\alpha_k' - \alpha}\right) \tag{3.1}$$

Here α_k and α_k' are given by the relations (2.2). A direct verification shows that the series (3.1) is convergent. Integrating the equation (3.1), we find

$$h_-(\alpha) = -\frac{1}{2}\ln(\alpha - iv) + \ln(\alpha - \alpha_0') + \tag{3.2}$$

$$\sum_{k=1}^{\infty} [\ln(\alpha - \alpha_k') - \ln(\alpha - \alpha_k) - \delta_k] + c$$

The constants δ_k must be chosen in such a way that the series (3.2) converges (c is an arbitrary constant). It is not difficult to show that convergence is ensured for $\delta_k = (4k)^{-1}$. Another choice of δ_k is equivalent to multiplication by a constant.

Taking Eq. (2.5) into account, we obtain an expression for $g_-(\alpha)$ in the form of an infinite product, namely,

$$g_-(\alpha) = \exp h_-(\alpha) = e^c (\alpha - iv)^{-1/2} (\alpha - \alpha_0') \Pi \tag{3.3}$$

$$\Pi = \Pi(\beta) = \prod_{k=1}^{\infty} \frac{t_k' + \beta}{t_k + \beta} \exp\left(-\frac{1}{4}k\right), \quad \beta = \frac{i\alpha}{\sqrt{3\pi V}}$$

The relations (1.8), (1.16), (2.8) and (3.3) yield a complete solution of the problem (1.1), (1.2) in the form of a single complex integral.

In what follows we shall need expansions of $\Pi^{-1}(\beta)$ for $|\beta| \ll 1$ and $|\beta| \gg 1$

$$\ln \Pi^{-1}(\beta) = \ln \Pi^{-1}(0) + \gamma_1\beta + \gamma_2\beta^2 + \ldots + \gamma_n\beta^n + \ldots \tag{3.4}$$

$$\Pi^{-1}(\beta) = e^{\gamma/4} \beta^{1/2}\left[1 - \frac{d}{\beta} - \frac{1}{36\pi^2}\frac{\ln\beta}{\beta^2} + O\left(\frac{1}{\beta^2}\right)\right] \tag{3.5}$$

The constants γ_n and d are defined by the relations

$$\gamma_n = \frac{(-1)^n}{n} \sum_{k=1}^{\infty} \frac{(t_k)^n - (t_k')^n}{(t_k t_k')^n}, \qquad d = \sum_{k=1}^{\infty}\left(t_k' - t_k - \frac{1}{4\sqrt{k}}\right) - \frac{1}{2} + \frac{1}{4}\zeta\left(\frac{1}{2}\right)$$

Here γ is Euler's constant and $\zeta(z)$ is the Riemann ζ-function. We note that obtaining the expansion (3.5) is not an entirely trivial problem.

Noting that $g_+(\alpha) = g(\alpha) / g_-(\alpha)$, we obtain

$$\frac{\partial\varphi}{\partial y}(x, 0) = \frac{-i}{2\pi}(-2iv)^{1/2} g_-(-iv) e^{-c} \int_{i\tau-\infty}^{i\tau+\infty} \frac{g(\alpha)(\alpha - iv)^{1/2} e^{-i\alpha x}}{(\alpha - \alpha_0')\Pi(\beta)(\alpha - iv)^{1/2}} d\alpha \tag{3.6}$$

With the aid of the relations (3.5) and (2.4) we can obtain an asymptotic expansion for the factor accompanying the exponential term in the integral (3.6) for $|\alpha| \gg 1$. Integrating this expansion termwise, we find (for $v \to 0$)

$$j(x) = -\frac{\partial\varphi}{\partial y}(x, 0) = e^{\gamma/4} t\Pi(0)\pi^{-1/2}x^{-1/2}(3\pi V)^{1/4} \times \tag{3.7}$$

$$\left[1 + 2(-t + d)u - \frac{u^2 \ln u}{27\pi^{5/2}} + O(u^2)\right], \quad u = \sqrt{3\pi V} x$$

We have thus obtained the first three terms of the expansion of $j(x)$ for $x \to +0$.

In order to obtain an expansion for large x, we deform the contour of integration in (3.6) in such a way as to envelope the cut in the lower half-plane from both sides. We then obtain (as $v \to 0$)

$$\frac{\partial\varphi}{\partial y}(x, 0) = \frac{\exp(-i\pi/6)}{2\pi}\left(\frac{2}{3\pi}\right)^{1/3} t\Pi(0)\sqrt{3\pi V}\int_0^{\infty} \frac{\exp(-u\xi)\,\xi^{-2/3}}{\xi + t}\Pi^{-1}(\xi)\psi(\xi)\,d\xi$$

$$\psi(\xi) = \mathrm{Ai}'\left[\exp\left(\frac{i\pi}{3}\right)\left(\frac{3\pi}{2}\right)^{2/3}\xi^{4/3}\right] \Big/ \mathrm{Ai}\left[\exp\left(\frac{i\pi}{3}\right)\left(\frac{3\pi}{2}\right)^{2/3}\xi^{4/3}\right] +$$

$$\exp\left(\frac{i\pi}{3}\right)\mathrm{Ai}'\left[\exp\left(-\frac{i\pi}{3}\right)\left(\frac{3\pi}{2}\right)^{2/3}\xi^{4/3}\right] \Big/ \mathrm{Ai}\left[\exp\left(-\frac{i\pi}{3}\right)\left(\frac{3\pi}{2}\right)^{2/3}\xi^{4/3}\right]$$

Using Watson's lemma in [7] and the expansion (3.4), we obtain finally an expression for the flow density

$$j(x) = -\frac{\partial \varphi}{\partial y}(x, 0) = \frac{(2V)^{1/3} 3^{1/3}}{\Gamma(1/3)} x^{-1/3}\Big[1 + \frac{d_1}{3}\frac{1}{u} + \frac{2A}{3}\frac{1}{u^{4/3}} + \qquad (3.8)$$

$$\frac{4d_2}{9}\frac{1}{u^2} + \frac{10}{9} d_1 A \frac{1}{u^{7/3}} + \frac{28}{27} d_3 \frac{1}{u^3} + \frac{20}{87} d_2 A \frac{1}{u^{10/3}} + O\left(\frac{1}{u^4}\right)\Big]$$

The constants A, d_1, d_2, d_3 are determined by the relations

$$A = 3\left(\frac{\pi}{2}\right)^{2/3}\Gamma^2\left(\frac{2}{3}\right) \Big/ \Gamma^2\left(\frac{1}{3}\right), \quad d_1 = \gamma_1 - \frac{1}{t}$$

$$d_2 = \gamma_2 + \frac{\gamma_1^2}{2} + \frac{\gamma_1}{t} + \frac{1}{t^2},$$

$$d_3 = \gamma_3 + \gamma_1\gamma_2 + \frac{\gamma_1^3}{6} - \frac{1}{t}\left(\gamma_2 + \frac{\gamma_1^2}{2}\right) + \frac{\gamma_1}{t^2} - \frac{1}{t^3}$$

The first term in the expression (3.8) coincides with the DBLA [8]. The terms which follow give corrections to this approximation, which are connected with the longitudinal diffusion. The expression (3.7) shows that the true character of behavior of the flow density $j(x) \sim V^{1/4}x^{-1/2}$ as $x \to +0$ differs from that given by the DBLA ($j(x) \sim V^{1/3}x^{-1/3}$). Transition between these two forms of asymptotics occurs in the region $x \sim V^{-1/2}$, i.e. $u \sim 1$.

We conclude by making several general remarks. Without changing the boundary conditions, we consider the more general equation

$$aVy^{n-2}\frac{\partial C}{\partial x} = \frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2}, \quad n \geqslant 2, \quad a > 0, \quad V > 0 \qquad (3.9)$$

Here, a is a number on the order of unity, and V is a parameter of the problem. Corresponding to Eq. (1.1), for example, we have $a = 2$, $n = 3$. In the problem (3.9), as in (1.1), there is no length scaling; changing over to the variables

$$\bar{x} = xV^m, \quad \bar{y} = yV^m, \quad m = \frac{1}{n-1}$$

we obtain an equation not containing V,

$$a\bar{y}^{n-2}\frac{\partial C}{\partial \bar{x}} = \frac{\partial^2 C}{\partial \bar{x}^2} + \frac{\partial^2 C}{\partial \bar{y}^2} \qquad (3.10)$$

Thus,

$$C = C(xV^m, \; yV^m), \quad j(x) = V^m f(xV^m) \qquad (3.11)$$

Assuming that in a neighborhood of the point of discontinuity in the boundary conditions $r = \sqrt{x^2 + y^2} = 0$ for any V, the character of the solution of (3.9) is determined by the highest derivatives, we obtain the first term in the expansion of a bounded solution in a series in powers of r, namely,

$$C_0 = Ar^{1/2}\sin(^1/_2\varphi) \qquad (3.12)$$

So far, we have no knowledge of the constant A. The approximation (3.12) yields for $j(x)$ the result $j(x) \sim x^{-1/2}$. Then, taking the relations (3.11) into account, we have

$$j(x) \sim V^m (xV^m)^{-1/2} \qquad (3.13)$$

We see then that for $n = 3$ this expression coincides with the first term of the expansion (3.7) to within a constant of order unity. Of course, it also follows directly from the scaling invariance of the relations (3.11) that $A \sim V^{m/2}$. Discarding the convective term is most natural if use of the equation is made in the form (3.10). This indicates the true nature of the approximation (3.12): it is suitable when $\bar{r} = rV^m \ll 1$. We note that these considerations confirm the assumption (1.12), which we made earlier in connection with the integrable nature of the singularity of $j(x)$ as $x \to +0$.

For the Eq. (3.9) the DBLA reduces to discarding the term $\partial^2 C / \partial y^2$. Then $C(0, y) = 1$ and

$$C(x, y) = \Gamma^{-1}\left(\frac{1}{n}\right)\gamma\left(\frac{1}{n}, \frac{a}{n^2}\frac{Vy^n}{x}\right)$$

where $\gamma(\alpha, z)$ is the incomplete gamma-function. For the flow density the DBLA yields the result

$$j(x) = \frac{n}{\Gamma(1/n)}\left(\frac{a}{n^2}\right)^{1/n}\left(\frac{V}{x}\right)^{1/n} \qquad (3.14)$$

For the case $n = 3$, $a = 2$ this expression coincides with the first term of the expansion (3.8). We note that for $n = 2$ (piston profile) the approximation (3.14) has the same form as the approximation (3.13). This explains why the DBLA gives an exact expression for the flow density for a piston profile. To see that this is so, it is sufficient to equate the expression (3.14) to the expression obtained from equation (2.9) with the replacement $v \to V$.

The general relationship (3.13) means that, for the boundary conditions chosen, $j(x) \sim x^{-1/2}$, independently of the form of the velocity profile (of the choice of n) as $x \to +0$; and it is only the power to which V enters into this expression that depends on the form of the profile.

It would be interesting to apply the method of matched asymptotic expansions to the problem (1.1), (1.2). However, the results given here show that in an arbitrary case we cannot join the DBLA with the first term (3.12) of a direct coordinate expansion. We note, as B.A. Kupershmidt has shown, that Laplace's equation has the solution ($C \sim r^{2/3}\sin(^2/_3\varphi)$), which matches with the DBLA for $x > 0, \varphi \sim 0$. However, this solution gives an incorrect asymptotics $j(x)$ as $x \to +0$ and does not satisfy the boundary condition $\partial C(x, 0)/\partial y = 0$, $x < 0$. Moreover, the character of the exact answers (3.7), (3.8), in particular, the form of the constants appearing in them, compels one to doubt that they could be obtained by any simple approximate method.

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